9. FOKKER-PLANCK EQUATION OF SLOWING DOWN

Here, we find the distribution function in non-local coordinates, (v,ζ,ψ_x), meaning that particles are assigned to an orbit, rather than the distribution function being given at a local point in configuration- and velocity-space. It is then possible to describe the evolution of the orbits (e.g. slowing down), assuming the slowing-down time, τ_s, is long compared to a bounce time, τ_b. Beginning with the drift-kinetic equation, and expanding in multiple time scales as done in Ref. 18, we obtain a bounce-averaged, drift-kinetic equation on the τ_s time-scale:

$$\frac{\partial f(v,\zeta,\psi_x)}{\partial t} = \frac{1}{\tau_b} \oint d\tau \; \{C[\psi;f(v,\zeta,\psi_x)] - \frac{e}{m}\vec{E} \cdot \frac{\partial f}{\partial \vec{v}} + s\} \tag{9.1}$$

The bounce average on the RHS of (9.1) includes all the large banana-width effects. (The details of evaluating these bounce integrals are described in Appendix E.) Here, C is the Fokker-Planck operator, which is time-averaged around an entire bounce orbit (i.e. bounce averaged), and s is the fast ion source. The particle flows in (9.1) arise from collisions and from the electric field, $\vec{E}$, associated with the ohmic heating current. The collision operator, C(f), is a function of the local variables, $(R,z,v,\eta) \leftrightarrow (B,\psi,v,\eta)$, while f is labelled non-locally by the COM variables (v,ζ,ψ_x). To use (9.1), it must be transformed into the COM space.

9.1 Transformation of Collision Operator

The Fokker-Planck operator used here describes fast ion binary encounters with the background plasma. Fast ion interactions with themselves are neglected since the fp density is much less than the background density. Further, assuming azimuthal velocity-space symmetry for a $1/r^2$ force-law allows the Rosenbluth-MacDonald-Judd form [87] of the collision operator to be written as:

$$C(f) = \frac{-1}{v^2}\frac{\partial}{\partial v}\left[f\left(v^2\frac{\partial H}{\partial v}+\frac{\partial G}{\partial v}\right)\right] + \frac{1}{2v^2}\frac{\partial}{\partial v^2}\left(v^2 f\frac{\partial^2 G}{\partial v^2}\right)$$

$$+\frac{1}{2v^3}\frac{\partial G}{\partial v}\frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial f}{\partial \eta}\right] , \qquad (9.2)$$

where G and H are the Rosenbluth potentials:

$$G = \sum_\alpha \Gamma_\alpha \int d^3v'|\vec{v}-\vec{v}'|f_\alpha(v'), \qquad (9.3)$$

$$H = \sum_\alpha \Gamma_\alpha \frac{m_\alpha+m}{m_\alpha}\int d^3v' f_\alpha(v')/|\vec{v}-\vec{v}'| . \qquad (9.4)$$

The index, α, indicates a sum over all background species, with

$$\Gamma_\alpha = \frac{Z_\alpha^2 Z^2 e^4 \ell n\Lambda}{4\pi\varepsilon_o^2 m^2} . \qquad (9.5)$$

The local derivatives in (9.2) must be transformed to COM coordinates using the chain rule:

$$\left.\frac{\partial}{\partial \eta}\right|_{Rzv} = \left.\frac{\partial}{\partial \eta}\right|_{B\psi v} = \left.\frac{\partial \zeta}{\partial \eta}\right|_{B\psi v} \left.\frac{\partial}{\partial \zeta}\right|_{\psi \psi_x v} + \left.\frac{\partial \psi_x}{\partial \eta}\right|_{B\psi v} \left.\frac{\partial}{\partial \psi_x}\right|_{\psi v \zeta} \tag{9.6}$$

$$\left.\frac{\partial}{\partial v}\right|_{Rz\eta} = \left.\frac{\partial}{\partial v}\right|_{B\psi \eta} = \left.\frac{\partial}{\partial v}\right|_{\psi \psi_x \zeta} + \left.\frac{\partial \zeta}{\partial v}\right|_{B\psi \eta} \left.\frac{\partial}{\partial \zeta}\right|_{\psi \psi_x v} + \left.\frac{\partial \psi_x}{\partial v}\right|_{B\psi \eta} \left.\frac{\partial}{\partial \psi_x}\right|_{\psi v \zeta} , \tag{9.7}$$

where the subscripts indicate the variables held constant during differentiation.

Substituting (9.6) - (9.7) and (8.2) into (9.2) gives:

$$\begin{aligned} C(f) = {} & \frac{-1}{v^2} \frac{\partial}{\partial v} \left[f \left(v^2 \frac{\partial H}{\partial v} + \frac{\partial G}{\partial v} \right) \right] + \frac{1}{2v^2} \frac{\partial^2}{\partial v^2} \left(v^2 f \frac{\partial^2 G}{\partial v^2} \right) \\ & + \frac{1}{2v^3} \frac{\partial G}{\partial v} \left(\frac{\partial \zeta}{\partial \eta} \right)^2 \frac{B}{B_x} \frac{\partial}{\partial \zeta} \left[(1-\zeta^2) \frac{\partial f}{\partial \zeta} \right] \\ & + A_1 \frac{\partial f}{\partial \zeta} + A_2 \frac{\partial f}{\partial \psi_x} \\ & + A_3 \frac{\partial^2 f}{\partial \zeta \partial \psi_x} + A_4 \frac{\partial^2 f}{\partial \psi_x^2} + A_5 \frac{\partial^2 f}{\partial \zeta^2} \\ & + \frac{1}{v^2} \left[\frac{\partial \zeta}{\partial v} \frac{\partial}{\partial v} \left(v^2 \frac{\partial^2 G}{\partial v^2} \frac{\partial f}{\partial \zeta} \right) + \frac{\partial \psi_x}{\partial v} \frac{\partial}{\partial v} \left(v^2 \frac{\partial^2 G}{\partial v^2} \frac{\partial f}{\partial \psi_x} \right) \right] , \end{aligned} \tag{9.8}$$

where the coefficients A_i are:

$$A_1 = -\frac{1}{v^2}\frac{\partial\zeta}{\partial v}\left(v^2\frac{\partial H}{\partial v} + \frac{\partial G}{\partial v}\right) + \tfrac{1}{2}\frac{\partial^2 G}{\partial v^2}\left(\frac{\partial\zeta}{\partial v}\frac{\partial}{\partial\zeta}\frac{\partial\zeta}{\partial v} + \frac{\partial\psi_x}{\partial v}\frac{\partial}{\partial\psi_x}\frac{\partial\zeta}{\partial v} + \frac{\partial}{\partial v}\frac{\partial\zeta}{\partial v}\right)$$

$$+ \frac{1}{2v^3}\frac{\partial G}{\partial v}(1-\zeta^2)\left(\frac{\partial\zeta}{\partial\eta}\frac{\partial}{\partial\zeta} + \frac{\partial\psi_x}{\partial\eta}\frac{\partial}{\partial\psi_x}\right)\left(\frac{B}{B_x}\frac{\partial\zeta}{\partial\eta}\right) \tag{9.9}$$

$$A_2 = -\frac{1}{v^2}\frac{\partial\psi_x}{\partial v}\left(v^2\frac{\partial\psi}{\partial v} + \frac{\partial G}{\partial v}\right) + \tfrac{1}{2}\frac{\partial^2 G}{\partial v^2}\left(\frac{\partial\zeta}{\partial v}\frac{\partial}{\partial\zeta}\frac{\partial\psi_x}{\partial v} + \frac{\partial\psi_x}{\partial v}\frac{\partial}{\partial\psi_x}\frac{\partial\psi_x}{\partial v} + \frac{\partial}{\partial v}\frac{\partial\psi_x}{\partial v}\right)$$

$$+ \frac{1}{2v^3}\frac{\partial G}{\partial v}\left[(1-\zeta^2)\left(\frac{\partial\zeta}{\partial\eta}\frac{\partial}{\partial\zeta} + \frac{\partial\psi_x}{\partial\eta}\frac{\partial}{\partial\psi_x}\right)\left(\frac{B}{B_x}\frac{\partial\psi_x}{\partial\eta}\right)\right. \tag{9.10}$$

$$\left. -2\left(\frac{\partial\zeta}{\partial\eta}\frac{\partial\psi_x}{\partial\eta}\frac{B}{B_x}\right)\zeta\right],$$

$$A_3 = \frac{\partial\zeta}{\partial v}\frac{\partial\psi_x}{\partial v}\frac{\partial^2 G}{\partial v^2} + \frac{1}{v^3}\frac{\partial\zeta}{\partial\eta}\frac{\partial\psi_x}{\partial\eta}\frac{B}{B_x}(1-\zeta^2)\frac{\partial G}{\partial v}, \tag{9.11}$$

$$A_4 = \tfrac{1}{2}\left(\frac{\partial\psi_x}{\partial v}\right)^2\frac{\partial^2 G}{\partial v^2} + \frac{1}{2v^3}\left(\frac{\partial\psi_x}{\partial\eta}\right)^2\left(\frac{B}{B_x}\right)(1-\zeta^2)\frac{\partial G}{\partial v}, \tag{9.12}$$

$$A_5 = \tfrac{1}{2}\left(\frac{\partial\zeta}{\partial v}\right)^2\frac{\partial^2 G}{\partial v^2}. \tag{9.13}$$

The derivatives in the chain rule are found by taking $\left.\frac{\partial}{\partial\eta}\right|_{B\psi v}$ and $\left.\frac{\partial}{\partial v}\right|_{B\psi\eta}$ of (8.1) and (8.2), then solving the resulting pairs of 2 equations in 2 unknowns for $\partial\zeta/\partial\eta$, $\partial\psi_x/\partial\eta$, $\partial\zeta/\partial v$ and $\partial\psi_x/\partial v$:

$$\frac{\partial\zeta}{\partial\eta} = \frac{\left(\frac{B_x^2}{F_x}\right)\left\{\frac{B'_{ex}F}{B_x B}(1-\zeta^2) + \frac{2F_x}{F}\left[\frac{e}{mv\gamma} - \zeta\left(\frac{F_x}{B_{ex}}\right)'\right]\left[\zeta - \frac{G_x c}{v\gamma}\left(1-\frac{\psi}{\psi_x}\right)\right]\right\}}{(1+\zeta^2)(B'_{ex}-B'_{ox})}, \quad (9.14)$$

$$\frac{\partial\psi_x}{\partial\eta} = \frac{2B_x^2}{BF_x}\left\{\frac{F_x^2 B}{FB_x}\left[\zeta - \frac{G_x c}{v\gamma}\left(1-\frac{\psi}{\psi_x}\right)\right] - F\zeta\right\} \Big/ (1+\zeta^2)(B'_{ex}-B'_{ox}), \quad (9.15)$$

$$\frac{\partial\zeta}{\partial v} = \frac{-G_x c\gamma B'_{ex}}{v^2(B'_{ex}-B'_{ox})}\left(\frac{1-\zeta^2}{1+\zeta^2}\right)\left(1-\frac{\psi}{\psi_x}\right), \quad (9.16)$$

$$\frac{\partial\psi_x}{\partial v} = \frac{2G_x c\gamma B_x}{v^2(B'_{ex}-B'_{ox})}\left(\frac{\zeta}{1+\zeta^2}\right)\left(1-\frac{\psi}{\psi_x}\right). \quad (9.17)$$

The result of this transformation into the COM space is a 3-dimensional, second-order, partial differential equation (PDE) with only the coefficients being bounce-averaged. In contrast, without bounce-averaging, the result is a 4-dimensional, second-order, PDE.

9.2 Simplifying Assumptions

To examine 3.5-Mev alpha particle slowing-down, several additional approximations are possible. Speed-diffusion is important for energies above the initial alpha energy and when:

$$E_\alpha/E_i \lesssim (m\ /m_e)^{1/3}(m\ /m_i)^{2/3}Z_{eff}{}^{2/3} = 26.7\ Z_{eff}{}^{2/3} \tag{9.18}$$

Both limits are unimportant for plasma heating by alphas so speed-diffusion is neglected. In fact, Cordey [65] finds the correction due to speed-diffusion to be $\sim T_i/E_o$, which is a negligible correction ($< 10^{-3}$) in this problem. Charge-exchange losses due to impurities could be significant [88], however the reactor-grade plasmas of interest here require that Z_{eff} (and thus such impurities) be minimized: $Z_{eff} \sim 1$-2. Consequently, charge exchange losses are neglected because the cross sections are very low for MeV alphas in hydrogenic plasmas. Also, the pitch angle scattering term is small for low Z_{eff} and $v > v_c$, where v_c is the speed at which the slowing drag due to ions equals that due to electrons, with $E_c = \frac{1}{2}\, mv_c^2 \sim 32.1\ T_e$ for α's in a 50-50 D-T plasma (c.f. eqn. 4.13 of Ref. 18). For these purposes, by the time alphas have slowed to $E \lesssim E_c$, 90% of their energy has been deposited in the plasma, so the scattering contribution is also dropped. Petrie and Miley [8] have explicitly evaluated the contribution due to pitch-angle scattering and found it to be small. Electric field effects are important only for circulating particles after a large number of bounce periods: $N_b \sim \tau_s/\tau_b \sim 10^5$. This corresponds to energies, $N_b e\Phi \lesssim 10^5$ eV, where Φ is the toroidal loop potential driving the ohmic current, which is also insignificant for heating calculations. The simplified form describing the alpha slowing-down becomes:

$$\frac{\partial f}{\partial t} = \frac{1}{v^2}\left(\frac{\partial}{\partial v}\,\tilde{U}f + \widetilde{U\,\frac{\partial \zeta}{\partial v}}\,\frac{\partial f}{\partial \zeta} + \widetilde{U\,\frac{\partial \psi_x}{\partial v}}\,\frac{\partial f}{\partial \psi_x}\right) + \tilde{\mathcal{S}} \quad , \tag{9.19}$$

where $U = -\left(v^2\,\frac{\partial H}{\partial v} + \frac{\partial G}{\partial v}\right)$, and the bounce-average, $\tilde{D}$, of some quantity, D, is $\tilde{D} = \frac{1}{\tau_b}\oint d\tau D$. While low Z_{eff} was assumed in deriving this last equation, all the terms in (3.19) are rigorously independent of Z_{eff}. The resulting 3-dimensional first-order partial differential equation, together with the boundary conditions, in (v,ζ,ψ_x) space is straight-forward to solve.